Properness for iterations with uncountable supports

based on joint works of
Andrzej Rosłanowski and Saharon Shelah

presented by AR

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We need iterated forcing for λ !

The development of *Set Theory of the Reals* in the XX century included but was not restricted to

- explosion of Descriptive Set Theory,
- interest in small and/or pathological sets on the real line,
- the rise of the language of cardinal coefficients and Forcing Axioms.

All three stimulated and fed on the progress in the theory of forcing iterated with finite or countable supports. For instance one of the reasons that in 2000 Mathematics Subject Classification we have

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What has been happening in the recent studies of spaces like ${}^{\lambda}\lambda$ or ${}^{\lambda}2$ parallels the past developments in the Set Theory of the Reals. There is a substantial activity in all corresponding directions "for λ–reals" and **this gives a strong push for development of forcing iterated with uncountable supports.**

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Some notation

Before we continue we should fix some notation and terminology. From now on,

- λ is an uncountable cardinal satisfying $\lambda^{<\lambda} = \lambda$.
- In forcing, $p \leq q$ means that "q is stronger than p".
- Every forcing notion $\mathbb{P}$ is atomless and has the unique weakest element $\emptyset_{\mathbb{P}}$.
- By "λ–support iterations" we mean iterations in which domains of conditions are of size $\leq \lambda$. However, we will pretend that conditions in a λ–support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \zeta \rangle$ are total functions on ζ and for $p \in \mathbb{P}_\zeta$ and $\xi \in \zeta \setminus \mathrm{dom}(p)$ we will stipulate $p(\xi) = \underset{\sim}{\emptyset}_{\underset{\sim}{\mathbb{Q}}_\xi}$.

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Standard properness

Dealing with our λ–support iterations, we could start in the way suggested already in Shelah [Sh 100] and just repeat what has been done for CS iterations.

Definition 1

Let $\lambda = \lambda^{<\lambda}$. A notion of forcing $\mathbb{P}$ is said to be *λ–proper in the standard sense* if for all sufficiently large regular cardinals χ, there is some $x \in \mathcal{H}(\chi)$ such that whenever M is an elementary submodel of $\mathcal{H}(\chi)$ satisfying

$$|M| = \lambda, \qquad \mathbb{P}, x \in M \qquad M^{<\lambda} \subseteq M$$

and p is an element of $M \cap \mathbb{P}$, then there is a condition $q \geq p$ such that

$$q \Vdash \text{“}M[\underset{\sim}{G}_{\mathbb{P}}] \cap \mathrm{Ord} = M \cap \mathrm{Ord}\text{”}.$$

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Theorem 2 (Folklore; cf. Hyttinen and Rautila [HyRa01, §3])

Assume $\lambda^{<\lambda} = \lambda$ is an uncountable cardinal.

1. *If a forcing notion $\mathbb{P}$ is either strategically $(\leq\lambda)$–complete or it satisfies the λ^+–chain condition, then $\mathbb{P}$ is λ–proper.*
2. *If $\mathbb{P}$ is λ–proper, $p \in \mathbb{P}$, $\underset{\sim}{A}$ is a $\mathbb{P}$–name for a set of ordinals and $p \Vdash |\underset{\sim}{A}| \leq \lambda$, then there are a condition $q \in \mathbb{P}$ stronger than p and a set B of size λ such that $q \Vdash \underset{\sim}{A} \subseteq B$.*
3. *If $\mathbb{P}$ is λ–proper, then*

   $$\Vdash_{\mathbb{P}} \text{“}(\lambda^+)^{\mathbf{V}} \text{ is a regular cardinal ”.}$$

   Moreover, if $\mathbb{P}$ is also strategically $(<\lambda)$–complete, then the forcing with $\mathbb{P}$ preserves stationary subsets of λ^+.

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Also chain condition results look similarly:

Theorem 3 (Folklore; cf. Eisworth [Ei03, Proposition 3.1])

Assume $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$, and let $\bar{\mathbb{P}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \lambda^{++} \rangle$ be a λ–support iteration such that for $i \leq \lambda^{++}$ the forcing $\mathbb{P}_i$ is λ–proper and $\Vdash_{\mathbb{P}_i}$ "$|\underset{\sim}{\mathbb{Q}}_i| \leq \lambda^+$"
Then

1. *$\mathbb{P}_{\lambda^{++}}$ satisfies the λ^{++}–chain condition, and*
2. *for each $i < \lambda^{++}$ the forcing notion $\mathbb{P}_i$ has a dense subset of size $\leq 2^{\lambda^+}$ and $\Vdash_{\mathbb{P}_i} 2^\lambda = \lambda^+$.*

More could be added here, see e.g., Johnstone [Jo08].

What is missing then? The Preservation Theorem!

Suppose you try to repeat the proof of the preservation of properness in CS iterations for λ–support iterations of λ–proper forcing notions. You take, say, Goldstern's *Tools* [Go] and you re-do Section 3 there for the new context. You will have no problems until *Preliminary Lemma 3.17*, in particular the same argument as in Lemma 3.16 works here for the successor stages.

But you will get stuck in *Induction Lemma 3.18* and you will face difficulties at limit stages of cofinality less than λ. Why? It is really inconvenient to diagonalize λ objects (our model N is of size λ) in less than λ steps!

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Example

Let $\mathcal{S}_{\lambda}^{\lambda^+} \stackrel{\text{def}}{=} \{\delta < \lambda^+ : \text{cf}(\delta) = \lambda\}$. Suppose that a sequence $\langle A_\delta, h_\delta : \delta \in \mathcal{S}_{\lambda}^{\lambda^+} \rangle$ is such that for each $\delta \in \mathcal{S}_{\lambda}^{\lambda^+}$:

(a) $A_\delta \subseteq \delta$, $\text{otp}(A_\delta) = \lambda$ and A_δ is a club of δ, and

(b) $h_\delta : A_\delta \longrightarrow 2$.

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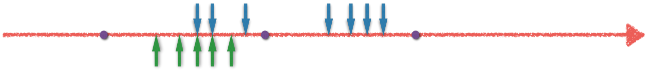

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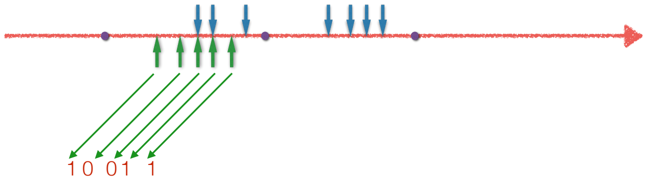

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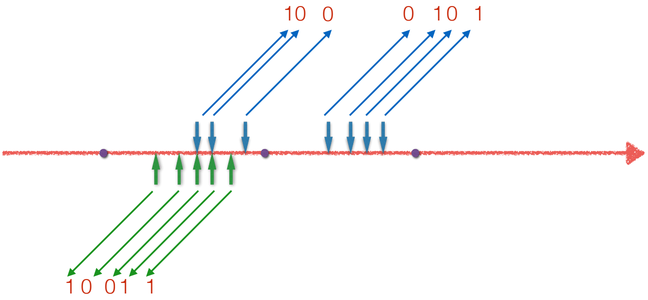

We define a forcing notion $\mathbb{Q}^* = \mathbb{Q}^*(\langle A_\delta, h_\delta : \delta \in \mathcal{S}_\lambda^{\lambda^+} \rangle)$ for adding a function $h : \lambda^+ \longrightarrow 2$ such that for every $\delta \in \mathcal{S}_\lambda^{\lambda^+}$ the set $\{\alpha \in A_\delta : h_\delta(\alpha) = h(\delta)\}$ contains a club of δ. A condition in the forcing is an approximation to h of size $<\lambda$. Thus:

a condition in $\mathbb{Q}^*$ is a tuple $p = (u^p, v^p, \bar{e}^p, h^p)$ such that

(a) $u^p \in [\lambda^+]^{<\lambda}$, $v^p \subseteq \mathcal{S}_\lambda^{\lambda^+} \cap u^p$,

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(c) if $\delta \in v^p$, then $\max(e_\delta^p) = \sup(u^p \cap \delta) > \sup(v^p \cap \delta)$,

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The order $\leq$ of $\mathbb{Q}^*$ is such that $p \leq q$ if and only if $u^p \subseteq u^q$, $h^p \subseteq h^q$, $v^p \subseteq v^q$, and each set e_δ^q is an end-extension of e_δ^p.

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Observation 4

The forcing notion $\mathbb{Q}^*$ is $(<\lambda)$–complete and $|\mathbb{Q}^*| = \lambda^+$. It satisfies the λ^+–chain condition, so it is also λ–proper.

If our λ is not inaccessible, $2^{\lambda^+} = \lambda^{++}$ and $2^\lambda = \lambda^+$, then some λ–support iterations of forcing notions like $\mathbb{Q}^*$ are not λ–proper, as a matter of fact this bad effect happens quite often!

Why? If λ–support iterations of forcings of type $\mathbb{Q}^*$ were λ–proper, we could use Theorem 3 and a suitable bookkeeping device to build a forcing notion forcing "$\lambda = \lambda^{<\lambda}$ is not inaccessible and the uniformization for colorings on lader systems holds true". However, this is not possible:

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Theorem 5 (Shelah [Sh:b], [Sh:f, Appendix, Theorem 3.6(2)])

Assume $\theta < \lambda = \mathrm{cf}(\lambda)$, $2^\theta = 2^{<\lambda} = \lambda$. Furthermore suppose that for each $\delta \in \mathcal{S}^{\lambda^+}_\lambda$ we have a club A_δ of δ. Then we can find a sequence $\langle d_\delta : \delta \in \mathcal{S}^{\lambda^+}_\lambda \rangle$ of colourings such that

- *$d_\delta : A_\delta \longrightarrow 2$ and*
- *for any $h : \lambda^+ \longrightarrow \{0, 1\}$ for stationarily many $\delta \in \mathcal{S}^{\lambda^+}_\lambda$, the set $\{i \in A_\delta : d_\delta(i) \neq h(i)\}$ is stationary in A_δ.*

Often λ–support iterations do work

Many positive results concerning not collapsing cardinals in iterations with uncountable supports are presented in literature. For instance:

- Kanamori [Ka80] considered iterations of λ–Sacks forcing notion and he proved that under some circumstances these iterations preserve λ^+.
- Fusion properties of iterations of tree–like forcing notions were used in Friedman and Zdomskyy [FrZd10] and Friedman, Honzik and Zdomskyy [FrHoZd13].
- Eisworth [Ei03] introduced a strong properness property and showed a preservation theorem for it.
- In [Sh 587] and [Sh 667] Shelah introduced several variants of *strong completeness/properness* and proved that they can be iterated. Those results generalized the preservation of "*S*–complete proper" in CS (and not adding new reals).

Our Program

In a series of articles [RoSh 655, RoSh 860, RoSh 777, RoSh 888, RoSh 890, RoSh:942, RoSh 1001] Shelah and AR try to isolate pairs of properties $\mathbf{P}^A_\lambda$ and $\mathbf{P}^B_\lambda$ of strategically $({<}\lambda)$–complete forcing notions such that

- $\mathbf{P}^B_\lambda(\mathbb{P})$ implies that $\mathbb{P}$ is λ–proper (so in particular forcing with $\mathbb{P}$ does not collapse λ^+),
- λ–support iterations of forcing notions with $\mathbf{P}^A_\lambda$ have the property $\mathbf{P}^B_\lambda$,
- all interesting forcings have one of the properties $\mathbf{P}^A_\lambda$.

While it is tempting, the requirement that one pair $(\mathbf{P}^A_\lambda, \mathbf{P}^B_\lambda)$ applies to *all* interesting forcing notions seems at the moment too much. But we are quite happy with the discovery of several of such pairs, each applying to a somewhat large class of forcings.

Of course, we would like to have *real preservation theorems,* i.e., $\mathbf{P}^A_\lambda = \mathbf{P}^B_\lambda$, but we can live without them.

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Note: for each property $\mathbf{P}_\lambda^A$ we could formulate the corresponding Forcing Axiom (and prove its consistency). Studying these axioms, their consequences and dependencies between them is the natural next step (left for the next generation).

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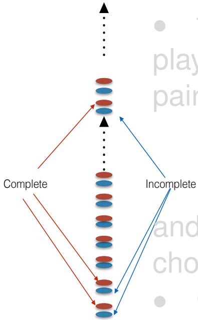
What is the strategic completeness?

Let $\mathbb{P}$ be a forcing notion. For an ordinal α, let $\partial_0^\alpha(\mathbb{P})$ be the following game of two players, *Complete* and *Incomplete*:



- the game lasts at most α moves and during a play the players **attempt** to construct a sequence $\langle (p_i, q_i) : i < \alpha \rangle$ of pairs of conditions from $\mathbb{P}$ in such a way that

$$(\forall j < i < \alpha)(p_j \leq q_j \leq p_i)$$

and at the stage $i < \alpha$ of the game, first Incomplete chooses p_i and then Complete chooses q_i.
- Complete wins if and only if for every $i < \alpha$ there are legal moves for both players.

- The forcing notion $\mathbb{P}$ is *strategically* $(<\lambda)$*–complete* (*strategically* $(\leq\lambda)$*–complete*, respectively) if Complete has a winning strategy in the game $\partial_0^\lambda(\mathbb{P})$ ($\partial_0^{\lambda+1}(\mathbb{P})$, respectively).

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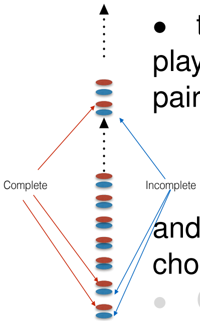

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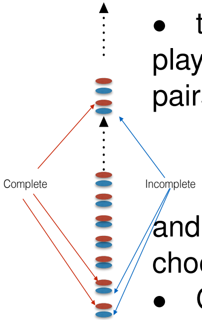

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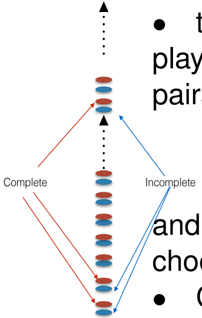

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Let us review some of the forcing notions that we will "cover" with our properties. We start with forcings in which conditions are complete λ–trees, i.e., $\vartriangleleft$–downward closed sets $T \subseteq {}^{<\lambda}\lambda$ in which every $\vartriangleleft$–chain of length $< \lambda$ has a $\vartriangleleft$–upper bound.

Suppose that $\bar{E} = \langle E_t : t \in {}^{<\lambda}\lambda \rangle$ is a system of $(<\lambda)$–complete filters on λ. We define forcing notions $\mathbb{Q}^{\ell,\bar{E}}$ for $\ell = 2, 3, 4$ as follows:

A condition in $\mathbb{Q}^{2,\bar{E}}$ is a complete λ–tree $T \subseteq {}^{<\lambda}\lambda$ such that

(a) if $t \in T$, then either $|\mathrm{succ}_T(t)| = 1$ or $\mathrm{succ}_T(t) \in E_t$, and

(b) $(\forall t \in T)(\exists s \in T)(t \vartriangleleft s \ \&\ |\mathrm{succ}_T(s)| > 1)$, and

(c)[2] if $j < \lambda$ and a sequence $\langle t_i : i < j \rangle \subseteq T$ is $\vartriangleleft$–increasing, $|\mathrm{succ}_T(t_i)| > 1$ for all $i < j$ and $t = \bigcup\limits_{i<j} t_i$, then $|\mathrm{succ}_T(t)| > 1$.

The order $\leq$ of $\mathbb{Q}^{2,\bar{E}}$ is the inverse inclusion, i.e., $T_1 \leq T_2$ if and only if $T_2 \subseteq T_1$.

Forcing notions $\mathbb{Q}^{3,\bar{E}}, \mathbb{Q}^{4,\bar{E}}$ are defined analogously, but the demand $(c)^2$ is replaced by the respective $(c)^\ell$:

$(c)^3$ for some club $C \subseteq \lambda$ of limit ordinals we have

$$(\forall t \in T)(\mathrm{lh}(t) \in C \Leftrightarrow |\mathrm{succ}_T(t)| > 1),$$

$(c)^4$ $(\forall t \in T)(\mathrm{root}(T) \vartriangleleft t \Rightarrow |\mathrm{succ}_T(t)| > 1)$.

$\diamond$ A natural special case of the forcing notions introduced above is when all filters E_t are club filters of λ. Then we omit $\bar{E}$ and call our forcing notions just $\mathbb{Q}^\ell$.

$\diamond$ The forcings $\mathbb{Q}^2, \mathbb{Q}^3$ and $\mathbb{Q}^4$ generalize the Miller forcing, the uniform Miller forcing and the Laver forcing, respectively.

$\diamond$ But note: we allow any complete filters E_t, they may be principal. Then

- if $E_t = \{\lambda\}$ for each $t \in {}^{<\lambda}\lambda$, then $\mathbb{Q}^{4,\bar{E}}$ is the λ–Cohen forcing $\mathbb{C}_\lambda$ and $\mathbb{Q}^{2,\bar{E}}$ generalizes the forcing notion $\mathbb{D}_\omega$ from Newelski and Rosłanowski [NeRo93],
- if for each $t \in {}^{<\lambda}\lambda$ we let E_t be the filter of all subsets of λ including $\{0, 1\}$, then the forcing notion $\mathbb{Q}^{2,\bar{E}}$ will be equivalent with Kanamori's λ–Sacks forcing of [Ka80, Definition 1.1].

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A related forcing notion is obtained if we introduce additional normal filter E on λ. Then $\mathbb{Q}^{1,\bar{E}}_{E}$ is defined as follows.

A condition p **in** $\mathbb{Q}^{1,\bar{E}}_{E}$ is a complete λ–tree $T \subseteq {}^{<\lambda}\lambda$ such that

- for every $t \in T$, either $|\mathrm{succ}_T(t)| = 1$ or $\mathrm{succ}_T(t) \in E_\nu$, and
- for every $\eta \in \lim_\lambda(T)$ the set $\{\alpha < \lambda : \mathrm{succ}_T(\eta{\restriction}\alpha) \in E_{\eta\restriction\alpha}\}$ belongs to E.

The order $\leq = \leq_{\mathbb{Q}^{1,\bar{E}}_{E}}$ is the reverse inclusion.

Observation 6

For $\bar{E}$ and E as above, the forcing notions $\mathbb{Q}^{1,\bar{E}}_{E}, \mathbb{Q}^{\ell,\bar{E}}$ (for $\ell \in \{2,3,4\}$) are $(<\lambda)$–lub–complete (i.e., increasing sequences of length $<\lambda$ have least upper bounds).

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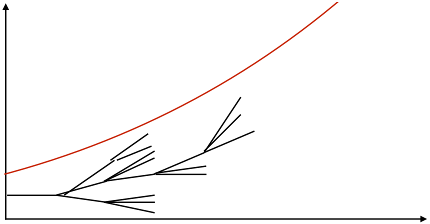
We may also consider "bounded" versions of the forcing notions introduced before. Assume that

- λ is weakly inaccessible, $\varphi : \lambda \longrightarrow \lambda$ is a strictly increasing function such that each $\varphi(\alpha)$ is a regular uncountable cardinal above α (for $\alpha < \lambda$),
- $\bar{F} = \langle F_t : t \in \bigcup\limits_{\alpha<\lambda} \prod\limits_{\beta<\alpha} \varphi(\beta)\rangle$ where F_t is a $<\varphi(\alpha)$–complete filter on $\varphi(\alpha)$ whenever $t \in \prod\limits_{\beta<\alpha} \varphi(\beta)$, $\alpha < \lambda$.



We define a forcing notion $\mathbb{Q}^2_{\varphi,\bar{F}}$ as follows.

A condition in $\mathbb{Q}^2_{\varphi,\bar{F}}$ is a complete λ–tree $T \subseteq \bigcup_{\alpha<\lambda} \prod_{\beta<\alpha} \varphi(\beta)$ such that

(a) for every $t \in T$, either $|\mathrm{succ}_T(t)| = 1$ or $\mathrm{succ}_T(t) \in F_t$, and

(b) $(\forall t \in T)(\exists s \in T)(t \vartriangleleft s \;\&\; |\mathrm{succ}_T(s)| > 1)$, and

(c)^2 if $j < \lambda$ and a sequence $\langle t_i : i < j \rangle \subseteq T$ is $\vartriangleleft$–increasing, $|\mathrm{succ}_T(t_i)| > 1$ for all $i < j$ and $t = \bigcup_{i<j} t_i$, then ($t \in T$ and) $|\mathrm{succ}_T(t)| > 1$.

The order of $\mathbb{Q}^2_{\varphi,\bar{F}}$ is the reverse inclusion.

Forcing notions $\mathbb{Q}^\ell_{\varphi,\bar{F}}$ for $\ell = 3, 4$ are defined similarly to $\mathbb{Q}^{\ell,\bar{E}}$.

Observation 7

For $\varphi, \bar{F}$ as above the forcing notions $\mathbb{Q}^{\ell}_{\varphi,\bar{F}}$ are strategically $(<\lambda)$–complete. Moreover, if $\bar{T} = \langle T_\alpha : \alpha < \delta \rangle \subseteq \mathbb{Q}^{\ell}_{\varphi,\bar{F}}$ is $\leq_{\mathbb{Q}^{\ell}_{\varphi,\bar{F}}}$–increasing and $\mathrm{root}(T_\alpha) \lhd \mathrm{root}(T_\beta)$ for $\alpha < \beta < \delta$, then $\bigcap\limits_{\alpha<\delta} T_\alpha \in \mathbb{Q}^{\ell}_{\varphi,\bar{F}}$ is the least upper bound to $\bar{T}$ and $\mathrm{root}(\bigcap\limits_{\alpha<\delta} T_\alpha) = \bigcup\limits_{\alpha<\delta} \mathrm{root}(T_\alpha)$.

A non-tree example

There are many interesting non-tree like forcing notions. For instance, consider the following generalization $\mathbb{P}^*$ of the forcing notion used by Goldstern and Shelah [GoSh 388]:

A condition in $\mathbb{P}^*$ is a pair $p = (\eta^p, C^p)$ such that $\eta^p : \lambda \longrightarrow \{-1, 1\}$ and C^p is a club of λ.

The relation $\leq = \leq_{\mathbb{P}^*}$ on $\mathbb{P}^*$ is defined by letting $p \leq q$ iff

1. $C^q \subseteq C^p$, $\eta^q \restriction \min(C^p) = \eta^p \restriction \min(C^p)$, and
2. for every successive members $\alpha < \beta$ of C^p we have

$$\big(\forall \gamma \in [\alpha, \beta)\big)\big(\eta^q(\gamma) = \frac{\eta^p(\alpha)}{\eta^q(\alpha)} \cdot \eta^p(\gamma)\big).$$

Observation 8

$\mathbb{P}^*$ *is a* $(<\lambda)$*–complete forcing notion of size* 2^λ.

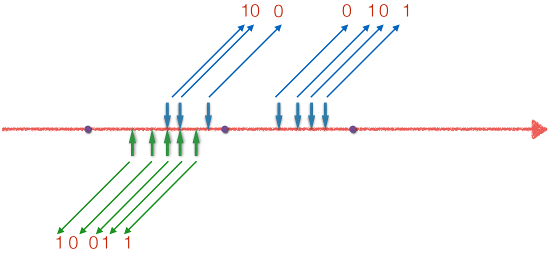
Bad forcing $\mathbb{Q}^*$ revisited

Remember the main counterexample $\mathbb{Q}^*$ to the preservation of λ–properness? We may modify it slightly and get it "covered"! Like before, $\langle A_\delta, h_\delta : \delta \in \mathcal{S}^{\lambda^+}_\lambda \rangle$ be such that

(a) $A_\delta \subseteq \delta$, $\mathrm{otp}(A_\delta) = \lambda$ and A_δ is a club of δ, and

(b) $h_\delta : A_\delta \longrightarrow 2$.

Also: let $\mathcal{S}'$ be an unbounded subset of the set of non-successor ordinals in λ such that $\mathcal{S} = \lambda \setminus \mathcal{S}'$ is stationary (and has a diamond)



The forcing notion $\mathbb{Q}^*_S$ is defined as follows:

a condition in $\mathbb{Q}^*_S$ is a tuple $p = (u^p, v^p, \bar{e}^p, h^p)$ such that

(a) $u^p \in [\lambda^+]^{<\lambda}$, $v^p \in [\mathcal{S}^{\lambda^+}_\lambda]^{<\lambda} \cap u^p$,

(b) $\bar{e}^p = \langle e^p_\delta : \delta \in v^p \rangle$, where each e^p_δ is a closed bounded subset of A_δ, and $e^p_\delta \subseteq u^p$, and

(c) if $\delta \in v^p$, then $\max(e^p_\delta) = \sup(u^p \cap \delta) > \sup(v^p \cap \delta)$,

(d) $h^p : u^p \longrightarrow 2$ is such that for each $\delta \in v^p$ we have

$$h^p \restriction \{\alpha \in e_\delta : \mathrm{otp}(\alpha \cap e_\delta) \in \mathcal{S}'\} \subseteq h_\delta;$$

the order $\leq$ of $\mathbb{Q}^*_S$ is such that $p \leq q$ if and only if $u^p \subseteq u^q$, $h^p \subseteq h^q$, $v^p \subseteq v^q$, and for each $\delta \in v^p$ the set e^q_δ is an end-extension of e^p_δ.

What our properties/proofs look like?

The properties $\mathbf{P}^A_\lambda$ we consider are phrased in the language of games of length λ. These games are played by two players, called *Generic* and *Antigeneric*. A good forcing notion is the one for which Generic has always a winning strategy.

Proving that our properties "can be iterated", we play those games on each coordinate. To exemplify what this means let us show the following proposition.

Proposition 9

Suppose that $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \zeta \rangle$ *is a* λ*–support iteration of strategically* $({<}\lambda)$*–complete forcing notions. Then* $\mathbb{P}_\zeta$ *is strategically* $({<}\lambda)$*–complete.*

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Proof

⋆ A winning strategy **st** of Complete in $\Game_0^\alpha(\mathbb{P})$ is *regular* if it instructs Complete to play $\emptyset_{\mathbb{P}}$ as long as Incomplete plays $\emptyset_{\mathbb{P}}$.

⋆ Note that if Complete has a winning strategy, then she also has a regular winning strategy.

⋆ For $\xi < \zeta^*$ let $\underset{\sim}{\mathbf{st}}_\xi$ be a $\mathbb{P}_\xi$–name for a **regular** winning strategy of Complete in $\Game_0^\lambda(\underset{\sim}{\mathbb{Q}}_\xi)$.

⋆ Now consider the following strategy for Complete: suppose the players arrived at a stage $\alpha < \lambda$ of a play of $\Game_0^\alpha(\mathbb{P}_\zeta)$ and they constructed a sequence $\langle (p_i, q_i) : i < \alpha \rangle^\frown \langle p_\alpha \rangle$ of conditions from $\mathbb{P}_\zeta$.

Complete puts forward a condition q_α with domain (support) the same as that of p_α and such that for each $\xi < \zeta$, $q_\alpha \restriction \xi$ forces $q_\alpha(\xi)$ to be the answer by strategy $\underset{\sim}{\mathbf{st}}_\xi$ to the partial play $\langle (p_i(\xi), q_i(\xi)) : i < \alpha \rangle^\frown \langle p_\alpha(\xi) \rangle$. □

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$\star$ A winning strategy **st** of Complete in $\Game_0^\alpha(\mathbb{P})$ is *regular* if it instructs Complete to play $\emptyset_{\mathbb{P}}$ as long as Incomplete plays $\emptyset_{\mathbb{P}}$.

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Typically our games are played on each coordinate, but at any given stage only $< \lambda$ coordinates are "active" (i.e., we play our games on more and more coordinates, but always less than λ).

Sometimes we additionally use trees of conditions (especially if λ is inaccessible) — we will use them in Part II, but let me finish today's meeting with the definition.

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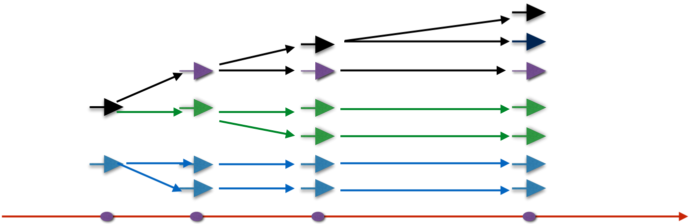
Trees of conditions

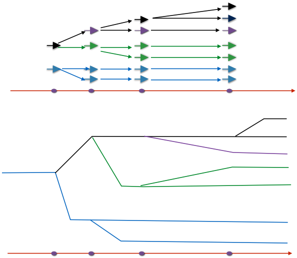
Let γ be an ordinal, $\emptyset \neq w \subseteq \gamma$. *A standard* $(w,1)^\gamma$*–tree* is a pair $\mathcal{T} = (T, \mathrm{rk})$ such that

- $\mathrm{rk} : T \longrightarrow w \cup \{\gamma\}$,
- if $t \in T$ and $\mathrm{rk}(t) = \varepsilon$, then t is a sequence $\langle (t)_\zeta : \zeta \in w \cap \varepsilon \rangle$,
- $(T, \lhd)$ is a tree with root $\langle\rangle$ and such that every chain in T has a $\lhd$–upper bound it T,
- if $t \in T$, then there is $t' \in T$ such that $t \trianglelefteq t'$ and $\mathrm{rk}(t') = \gamma$.



Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \gamma \rangle$ be an iteration.

$\diamond$ *A standard tree of conditions in* $\bar{\mathbb{Q}}$ is a system $\bar{p} = \langle p_t : t \in T \rangle$ such that



- (T, rk) is a standard $(w, 1)^\gamma$–tree for some $w \subseteq \gamma$,
- $p_t \in \mathbb{P}_{\mathrm{rk}(t)}$ for $t \in T$, and
- if $s, t \in T$, $s \lhd t$, then $p_s = p_t {\restriction} \mathrm{rk}(s)$.

$\diamond$ Let $\bar{p}^0, \bar{p}^1$ be standard trees of conditions in $\bar{\mathbb{Q}}$, $\bar{p}^i = \langle p^i_t : t \in T \rangle$. We write $\bar{p}^0 \leq \bar{p}^1$ whenever for each $t \in T$ we have $p^0_t \leq p^1_t$.

Theorem 10

Assume that $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \mathbb{Q}_i : i < \gamma \rangle$ *is a* λ*–support iteration such that for all* $i < \gamma$ *we have*

$$\Vdash_{\mathbb{P}_i} \text{“} \underset{\sim}{\mathbb{Q}}_i \text{ is strategically } ({<}\lambda)\text{–complete ”.}$$

Suppose that $\bar{p} = \langle p_t : t \in T \rangle$ *is a standard tree of conditions in* $\bar{\mathbb{Q}}$, $|T| < \lambda$, *and* $\mathcal{I} \subseteq \mathbb{P}_\gamma$ *is open dense. Then there is a standard tree of conditions* $\bar{q} = \langle q_t : t \in T \rangle$ *such that* $\bar{p} \leq \bar{q}$ *and* $(\forall t \in T)(\mathrm{rk}(t) = \gamma \ \Rightarrow \ q_t \in \mathcal{I})$.

Sometimes we are forced to deal with RS–conditions; they will be mentioned/explained in Part III.

Thank You!

Thank you for your attention today.
I hope you will come to the second part of this series — we will talk about various $^{\lambda}\lambda$*–bounding properties.*

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